A notes on alternative theories of gravity

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Contents

I. NOTATIONS USED

- In the present notes, we choose $M_{\rm pl} = 1$, $G = 1$, $\hbar = 1$ and $c = 1$.
- This unit system is called the reduced Planckian unit system, though $G = 1$ is often chosen in the plain Planckian unit system
- In this unit system, all quantities are defined in term of mass dimension.
  - $[m] = 1$, $[x] = -1$, $[t] = -1$,
  - Lagrangian $[\mathcal{L}] = 4$ (in a relativistic theory)
  - Therefore, for $\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$, $[\phi] = 1$

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- For fermions, from mass term $m\psi\bar{\psi} \to [\psi] = 3/2$.
- For a guage field $\mathcal{L}_{\rm gauge} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \to [F] = 2$ and $[A] = 1$, (from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$).
- Action $S = \int \sqrt{-g}\,\mathcal{L}$. Here $[S] = 0$ so $[d^4 x] = -4$

II. MOTIVATION BEHIND ALTERNATIVE THEORIES

In the current standard ΛCDM cosmological model, the gravitational interaction is described by Einstein's theory of General Relativity (GR). The main reason for this is perhaps related to its remarkable agreement with a wealth of precision tests of gravity done in the Solar System. These include the classical tests of gravitational redshift, the lensing of the light from background stars by the Sun and the anomalous perihelion of Mercury, as well as other tests such as the Shapiro time-delay effect measured by the Cassini spacecraft and Lunar laser ranging experiments which meausure the rate of change of the gravitational strength in the Solar system. Outside of the Solar system, GR is also in good agreement with the tests that involve changes in the orbital period of binary pulsars due to the emission of gravitational waves.

Despite of these tremendous successes, however, one can still think of a few reasons to expect/suspect/wish that GR does not provide us with the full picture. Therefore, there are some problems which can not be explained by ΛCDM model. They are as follows:

1. What is explaining the origin of the inflaton?
2. Nature of the Dark matter
3. - **Cosmological constant problem-*the magnitude problem*:** according to which the observed value of the cosmological constant is extravagantly small to be attributed to the vacuum energy of matter fields
   - **The coincidence problem:**: since there is just an extremely short period of time in the evolution of the universe in which the energy density of the cosmological constant is comparable with that of matter, why is this happening today when we are present to observe it?
4. **GR has no quantum limit:** all interactions must have a quantum field description, then GR cannot be the final answer and must be corrected. ⇒ **GR is not renormalizable and, therefore, cannot be conventionally quantized.**

These Problems make the concordance model more of an empirical fit to the data whose theoretical motivation can be regarded as quite poor. It is rather pointless to argue whether such a perspective would be better or worse than any of the other solutions already proposed. It is definitely a different way to address the same problems and, as long as these problems do not find a plausible, well-accepted, and simple solution, it is worth pursuing all alternatives.

III. GENERAL RELATIVITY IN A NUTSHELL

GR can be described by the Einstein-Hilbert action:

$$S = \int d^4x\,\sqrt{-g}\left[\frac{R}{16\pi G} - \mathcal{L}_{\rm matter}(\psi, g_{\mu\nu})\right]. \tag{1}$$

where g is the determinant of the 2-rank metric tensor field $g_{\mu\nu}$, R is called the Ricci scalar, $\mathcal{L}_{\rm matter}$ is the Lagrangian density that describes the forms of energy we know (dark matter, baryons, radiation, etc., described collectively by the field ψ) and the integration is taken over the whole four-dimensional x^μ. Einstein–Hilbert term in the standard theory is given by

$$\mathcal{L}_{\rm EH} = \sqrt{-g}\frac{1}{16\pi G}R \tag{2}$$

By varying this action w.r.t. $g_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T^{\rm matter}_{\mu\nu}. \tag{3}$$

Here $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ is the energy momentum tensor associated with $\mathcal{L}_{\rm matter}$. Covariant derivative of this equation will give

$$\nabla_\nu G^{\mu\nu} = 0 \Rightarrow \nabla_\nu T^{\mu\nu} = 0. \tag{4}$$

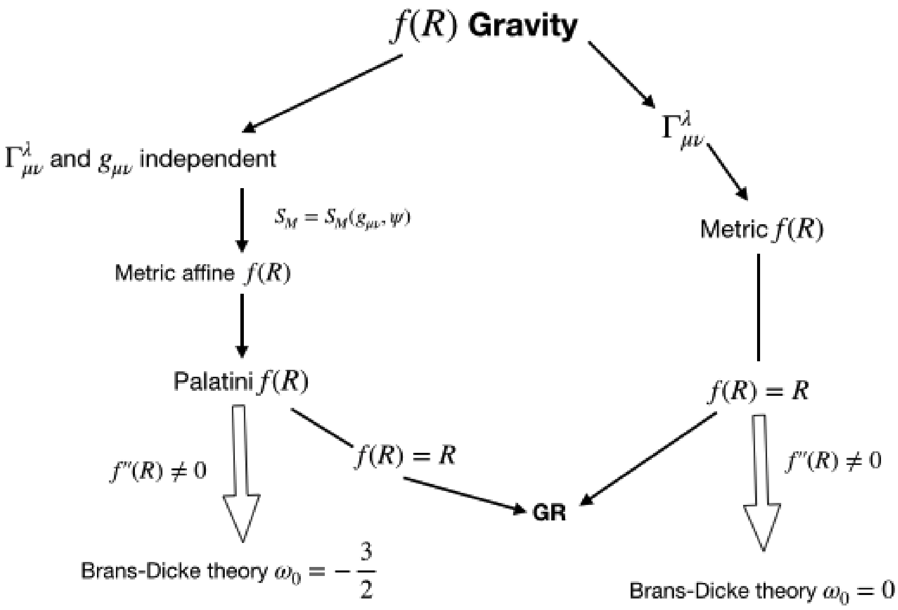

FIG. 1: $f(R)$ gravity

Now for a spatially homogeneous and isotropic universe, using the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{5}$$

where $k = 0$ for spatially flat universe. We get following equation using Einstein field equations (known as Friedmann equations):

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \tag{6}$$

$$2\dot{H} + 3H^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}\left(\rho + \frac{3p}{c^2}\right) \tag{7}$$

The two Friedmann Equations can be combined to yield the adiabatic equation

$$\frac{d}{dt}(\rho a^3 c^2) + p\frac{d}{dt}(a^3) = 0 \tag{8}$$

which is the relativistic version of the first law of thermodynamics: $T\, dS = DE + p\, dV = 0$, if the entropy is constant.

IV. IMPORTANT MODELS OF THE MODIFIED GRAVITY

The theory of gravity can be modified by many ways and their are very large number of works has been done. All these models are more or less describing the evolution history from inflation time to present time by indirect explanations. There are still numerous proposals for modified gravity in contemporary literature. The best-known alternative to GR are:

- Dvali- Gabadadze-Porrati gravity
- Brane- world gravity
- Tensor-vector-scalar theory
- Einstein-Aether theory

Some examples: Actions

Modification can be done by following ways-

- Minimally coupled with gravity

$$S = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}[R + \mathcal{L}_{\text{matter}}] \tag{9}$$

This represent simple theory with no dark energy and known as Einstein-Hilbert action. In above $\mathcal{L}_{\text{matter}}$ can be $\mathcal{L}_{\text{matter}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$ without dark energy. In case of dark energy,

$$S = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}[(R - 2\Lambda) + \mathcal{L}_{\text{matter}}] \tag{10}$$

This action represent the one of the model of the ΛCDM model. Which is well tested and also explained the large number of the observations.

- Non Minimally coupled with gravity like $f(\phi, R)$ or $f(\phi)$R kind of theories.

$$S = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}[f(\phi, R) + \mathcal{L}_{\text{matter}}] \tag{11}$$

Another way to modify the gravity theory is to modify the R part in the above all equations like done Starobinsky model.

$$S = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}[f(\phi)R + \mathcal{L}_{\text{matter}}] \tag{12}$$

in above two equations we can add Λ term to see the effect of dark energy. One example is $f(\phi, R) = \frac{1}{2}\xi\phi^2R$

- Or by modifying the

$$S = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}[f(R) + \mathcal{L}_{\text{matter}}] \tag{13}$$

In Starobinsky model $f(R) = R + R^2$. In more general their are models like $R + \phi^\alpha R^\beta$ Higgs inflation in $f(\phi, R)$ theory.

V. $f(R)$ GRAVITY: METRIC FORMALISM

The total action for the $f(R)$ gravity is:

$$S_{\text{total}} = \int d^4x\sqrt{-g}f(R) + S_{\text{matter}}(g_{\mu\nu}, \psi). \tag{14}$$

Here ψ collectively denotes the matter fields. Variation with respect to the metric gives, after some manipulations and modulo surface terms,

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(R) = \kappa T_{\mu\nu}, \tag{15}$$

where $\kappa = 8\pi G$, which can be rearranged to

$$G_{\mu\nu} = \frac{\kappa}{f'(R)}\left(T_{\mu\nu} + T^{\text{eff}}_{\mu\nu}\right). \tag{16}$$

Here

$$T^{\text{eff}}_{\mu\nu} = \frac{1}{\kappa}\left[\frac{1}{2}\left(f(R) - Rf'(R)\right)g_{\mu\nu} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right] \tag{17}$$

is an effective stress-energy tensor which does not have the canonical form quadratic in the first derivatives of the field $f'(R)$ but contains terms linear in the second derivatives. Also we define $G_{\text{eff}} = G/f'(R)$.

Following criteria must be satisfied in order $f(R)$ model to be theoretically consistent and compatible with cosmological observations and experiments:

- The model must have the correct cosmological dynamics
- Exhibit the correct behavior of gravitational perturbations
- generate cosmological perturbation compatible with the cosmological constraints from CMB, large scale structure formation, big bang nucleosynthesis and gravity waves.

Since FLRW metric is the best metric that explain the most of the observed phenomena, we have

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{18}$$

Therefore, we will have

$$H^2 = \frac{\kappa}{3f'}\left[\rho + \frac{Rf'-f}{2} - 3H\dot{R}f''\right] \tag{19}$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left[P + (\dot{R})^2 f''' + 2H\dot{R}\, f'' + \ddot{R}\, f'' + \frac{1}{2}(f - Rf')\right]. \tag{20}$$

We assume that $f' > 0$ in order to have a positive effective gravitational coupling and $f'' > 0$ to avoid the Dolgov-Kawasaki instability. We can define a effective energy density as

$$\rho_{\text{eff}} = \frac{Rf'-f}{2f'} - 3H\frac{\dot{R}f''}{f'}, \tag{21}$$

$$P_{\text{eff}} = \frac{(\dot{R})^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')}{f'} \tag{22}$$

Here ρ_{eff} has to be non-negative in a spatially flat FLRW space time in the limit of $\rho \to 0$ (see equation 19). Therefore, in vacuum, eq (19) and (20) can take the form of the standard Friedmann equation:

$$H^2 = \frac{\kappa}{3}\rho_{\text{eff}} \tag{23}$$

$$\frac{\ddot{a}}{a} - \frac{\kappa}{6}[\rho_{\text{eff}} + 3P_{\text{eff}}]. \tag{24}$$

Hence in vacuo, the curvature correction can be viewed as an effective fluid. It is thus equation of state is:

$$w_{\text{eff}} \equiv \frac{P_{\text{eff}}}{\rho_{\text{eff}}} = \frac{(\dot{R})^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')}{\frac{Rf'-f}{2} - 3H\dot{R}f''} \tag{25}$$

Since Denominator must be positive, the sign of w_{eff} is determined by the numerator. For a metric $f(R)$ model to mimic the de Sitter equation of state i.e. $w_{\text{eff}} = -1$, then

$$\frac{f'''}{f''} = \frac{\dot{R}H - \ddot{R}}{(\dot{R})^2}. \tag{26}$$

Example:
For $f(R) \propto R^n$ and $a(t) = a_0 \frac{t}{t_0}^{\alpha}$,

$$w_{\text{eff}} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \tag{27}$$

for $n \neq 1$,

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{28}$$

For $n = 2$, $w_{\text{eff}} = -1$ and $\alpha = \infty$ as ex- pected, considering that quadratic corrections to the Einstein-Hilbert Lagrangian were used in the well- known Starobinsky inflation.

VI. SCALAR TENSOR THEORY OF GRAVITY

- Theory of a scalar field (or fields) coupled to gravity.
- In theoretical physics, a scalar–tensor theory is a field theory that includes both a scalar field and a tensor field to represent a certain interaction.

A. Why Scalar-tensor theory?

- Einstein's general theory of relativity is a geometrical theory of spacetime. The fundamental building block is a metric tensor field. For this reason the theory may be called a "tensor theory."
- An "alternative theories" to the Einstein's general theory is scalar tensor theory.
- It is built on the solid foundation of general relativity, and the scalar field comes into play in a highly nontrivial manner, specifically through a "nonminimal coupling term," as will be explained shortly.
- The scalar–tensor theory was conceived originally by Jordan, who started to embed a four-dimensional curved manifold in five-dimensional flat space-time.
- In standard Einstein theory the space-time metric tensor, or more precisely geometry, is the sole quantity describing gravity. Dicke suggested the addition of another, scalar, field.
- The general Lagrangian for the scalar field living in four-dimensional curved space-time

$$\mathcal{L}_J = \sqrt{-g}\left[\varphi_J^m\left(R - \omega_J \frac{1}{\phi_J^2} g^{\mu\nu}\partial_\mu\varphi_J\partial_\nu\varphi_J\right) + L_{\text{matter}}(\varphi_J, \Psi)\right] \tag{29}$$

  here m and ω_j are two constant and ϕ_J is the scalar field in Jordan frame. Ψ_j represents the matter fields collectively.

  - $\varphi_J^m R \rightarrow$ give birth to the scalar tensor theory
  - The term $L_{\text{matter}}(\varphi_J, \Psi)$ is the matter Lagrangian, which depends generally on the scalar field, as well.
  - Second term resembles the kinetic term of the φ_J field.

- **Brans-Dicke model or prototype BD model (1961):** $\varphi = \varphi_J^m$

$$\mathcal{L}_{\text{BD}} = \sqrt{-g}\left[\varphi\left(R - \omega\frac{1}{\varphi^2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right) + L_{\text{matter}}(\Psi)\right] \tag{30}$$

  last term is now independent of φ. ω is the Brans–Dicke coupling constant.

- Now defining a new variable $\varphi = \frac{1}{2}\xi\phi^2$ and $\varepsilon\xi^{-1} = 4\omega$, equation (30), will reduced to

$$\mathcal{L}_{\text{BD}} = \sqrt{-g}\left[\frac{1}{2}\xi\phi^2 R - \frac{1}{2}\varepsilon g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + L_{\text{matter}}\right] \tag{31}$$

  Here $\varepsilon = \pm = \text{Sign}\omega$. Now no singularity appears. $\varepsilon = +1$ corresponds to a normal field having a positive energy, in other words no ghost particles. $\varepsilon = -1$ indicates a negative energy and is nonphysical.

B. Where does the scalar field come from?

They may arise from

- The scalar field arising from the size of compactified internal space
- The dilaton from string theory
- The scalar field in a brane world
- The scalar field in the assumed two-sheeted structure of space-time

C. Brans-Dicke model

In theoretical physics, the Brans–Dicke theory of gravitation (sometimes called the Jordan–Brans–Dicke theory) is a theoretical framework to explain gravitation. It is a competitor of Einstein's theory of general relativity. This scalar, ϕ, like gravity, has all matter as its source, and thus, in some semantic sense could be described as an extension of the gravitational field from purely geometric to geometric plus scalar, thus the term "scalar-tensor." The scalar aspect of ϕ is important since it cannot be "gauged" away as the metric and connection components can be, at least locally. This particular formalism was published by Brans and Dicke in 1961 (Brans and Dicke 1961).

Field equations

Varying equation (31) in with respect to $g^{\mu\nu}$, we will get following field equation:

$$2\varphi G_{\mu\nu} = T_{\mu\nu} + T^{\phi}_{\mu\nu} - 2\left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right)\varphi \tag{32}$$

$$\Box\varphi = \zeta^2 T \tag{33}$$

$$\nabla_\mu T^{\mu\nu} = 0. \tag{34}$$

Proof of equation (32) :
Equation (32) can be obtained by varying with respect to $g^{\mu\nu}$.
Proof of equation (33) :
In equation (33), $T = g^{\mu\nu}T_{\mu\nu}$. Also $T^{\phi}_{\mu\nu}$ is defined as

$$\frac{\delta(\sqrt{-g}L_\phi)}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\,T^{\phi}_{(\mu\nu)} \tag{35}$$

where $(\mu\nu)$ is defined following property: $A_{(\mu\nu)} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$. Lagrangian $L^{\phi} = -\frac{1}{2}\varepsilon g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Therefore,

$$T^{\phi}_{\mu\nu} = \varepsilon[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2]. \tag{36}$$

Here $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Equation (33) can obtained by varying equation (31) with respect to ϕ. Therefore, we will have:

$$\xi\phi R + \varepsilon\Box\phi = 0. \tag{37}$$

Here $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$. Trace of equation (32), and after some simplifications, we will get following relation:

$$-2\varphi R = T - \varepsilon(\partial\phi)^2 - 6\Box\varphi. \tag{38}$$

Using all these equations, we can obtain equation (33). Here $\zeta = 6 + 4\omega$. Equation (33) shows that the scalar field has its source given by the trace of the matter energy–momentum tensor.

The weak-field approximation

We try to see what physical results can be derived from these field equations. We first study the limit in which the fields are weak. This would correspond to the Newtonian approximation of Einstein's equation. We can write $\phi = v + Z\sigma(x)$, where v is a constant with the dimension of mass, corresponding to a "vacuum expectation value" in the quantum field theory. Also it will play the role of the "background field" in cosmology. Z is another constant. We assume that $v \gg Z\sigma$, to linearize the equations with respect to σ. Zeroth order of (31), gives $\sqrt{-g}\frac{1}{2}\xi v^2 R$. Since this has the same appearance as the Einstein–Hilbert term, therefore, we will have $\xi v^2 = M^2_{\rm pl} = 1 \Rightarrow v = \xi^{-1}$. Thus $\varphi = \frac{1}{2} + \xi^{1/2}Z\sigma$. Equation (33), will give

$$\Box\sigma = \xi^{-1/2}Z^{-1}\zeta^2\,T \tag{39}$$

In the Newtonian limit, $T = -\rho$ (ρ=mass density in the rest frame of the object). Therefore,

$$\nabla^2\,\sigma(\vec{r}) = -\xi^{-1/2}Z^{-1}\zeta^2\,\rho. \tag{40}$$

which looks like Poisson equation. For a point mass with the total mass M, solution is

$$\sigma(\vec{r}) = \frac{M}{M_{\rm pl}}\,\xi^{-1/2}Z^{-1}\zeta^2\left(\frac{1}{4\pi r}\right) \tag{41}$$

Linearization of metric

Using $g_{\mu\nu} = \eta_{\mu\nu} + M_{\rm pl}^{-1} h_{\mu\nu}(x)$. Then

$$G_{\mu\nu} = \frac{1}{2M_{\rm pl}}\left[-\Box h_{\mu\nu} + \partial_\mu\partial_\lambda h^\lambda_\nu + \partial_\nu\partial_\lambda h^\lambda_\mu - \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h)\right], \tag{42}$$

Here $h =^\mu_\mu$. Now using equations $\varphi = \frac{1}{2} + \xi^{1/2} Z\sigma$ and $h =^\mu_\mu$ in equation (32) and dropping terms of $\mathcal{O}(\sigma^2)$, we may put $T^\phi_{\mu\nu} \approx 0$. We will have following

$$\Box\phi^2 = 2vZ\Box\sigma \tag{43}$$
$$\nabla_\mu\nabla_\nu\phi^2 = 2vZ\partial_\mu\partial_\nu\sigma. \tag{44}$$

Thus equation (32) will give:

$$\xi\frac{v^2}{M_{\rm pl}}\left(-\Box h_{\mu\nu} + ...\right) = T_{\mu\nu} - 2vZ\xi(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\sigma. \tag{45}$$

By rearranging terms and eliminating v by $v = \xi^{-1/2}$, above equation will reduce to

$$\Box h_{\mu\nu} - \partial_\mu\partial_\lambda h^\lambda_\mu + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial_\sigma\partial_\rho h^{\sigma\rho} - \Box h) - 4Z\xi^{1/2}(\eta_{\mu\nu} - \partial_\mu\partial_\nu)\sigma = -2M_{\rm pl}^{-1}T_{\mu\nu}. \tag{46}$$

Last term on the left hand side represent mixing between two fields, $h_{\mu\nu}$ and σ, which can be removed by diagonalization using the following redefined fields

$$\chi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - 2Z\xi^{1/2}\eta_{\mu\nu}\sigma. \tag{47}$$

which can be inverted to

$$h_{\mu\nu} = \chi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\chi - 2Z\xi^{1/2}\eta_{\mu\nu}\sigma. \tag{48}$$

After imposing a coordinate condition $\partial_\lambda\chi^\lambda_\nu = 0$, we will obtain equation without mixing:

$$\Box\chi_{\mu\nu} = -\frac{2}{M_{\rm pl}}T_{\mu\nu}. \tag{49}$$

For a point mass at rest with mass M, solution is

$$\chi_{00}(\vec{r}) = 2\frac{M}{M_{\rm pl}}\frac{1}{4\pi r}, \tag{50}$$
$$\chi_{0i}(\vec{r}) = 0 \tag{51}$$
$$\chi_{ij}(\vec{r}) = 0. \tag{52}$$

Now let us consider another point mass with mass m, which moves in the fields $\chi_{\mu\nu}$ and σ. The equation of motion of a point mass m in the field $\chi_{\mu\nu}$ will be a geodesic. This implies that the prototype BD model falls into the category of "metric theories," in which equations of motion of matter are determined by the space-time metric. So gravitational potential is given by

$$V = -\frac{1}{2}\frac{m}{M_{\rm pl}}h_{00}. \tag{53}$$

But

$$h_{00} = \chi_{00} + \frac{1}{2}\chi + 2Z\chi^{1/2}\sigma = \frac{1}{2}\chi_{00} + 2Z\xi^{1/2}\sigma, \tag{54}$$

where $\chi = -\chi_{00}$. In this way we obtain

$$V = V_\chi + V_\sigma, \tag{55}$$

where

$$V_\chi = -\frac{1}{4}\frac{m}{M_{\rm pl}}\chi_{00} \Rightarrow V_\chi = -\frac{1}{2}\frac{mM}{M_{\rm pl}}\frac{1}{4\pi r} \quad (56)$$

$$V_\sigma = -\frac{1}{4}\frac{m}{M_{\rm pl}}Z\xi^{1/2}\sigma \Rightarrow V_\sigma = -\frac{1}{2}\frac{mM}{M_{\rm pl}}\zeta^2\frac{1}{4\pi r} \quad (57)$$

We notice that the portion V_χ from the tensor field and V_σ from the scalar field share exactly the same dependence on the masses and the distance. This implies that the force due to the scalar field couples to matter in proportion to the inertial mass of an object. It follows that any object falls with the common acceleration even with the contribution of the scalar force included. We can combine above two equations.

$$V = -G_1\frac{m^2}{r} \quad (58)$$

where $G_1 = \frac{4+2\omega}{3+2\omega}G$. When $\omega \to \infty$, $G_1 = G$. This is be- cause the scalar coupling to matter vanishes in this limit.

VII. THE HORNDESKI/GALILEON MACHINER

Over recent years, the so-called Horndeski/Galileon models have been gathering considerable attention in the community. The action of the Horndeski model is

$$S = \int d^4x\sqrt{-g}[\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 = \mathcal{L}_m(\psi, g_{\mu\nu})], \quad (59)$$

where

$$\mathcal{L}_2 = G_2(\varphi, X), \quad (60)$$

$$\mathcal{L}_3 = -G_3(\varphi, X)\Box\varepsilon \quad (61)$$

$$\mathcal{L}_4 = -G_4(\varphi, X)R + \frac{d}{dX}G_4(\varphi, X)[(\Box\varepsilon)^2 - (\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi)] \quad (62)$$

$$\mathcal{L}_5 = G_5G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi - \frac{1}{6}\frac{d}{dX}G_5(\varphi, X)\left[(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)(\nabla_\mu\nabla_\nu\varphi) + 2(\nabla_\mu\nabla^\nu\mathit{\Delta})(\nabla_\nu\nabla^\rho\varphi)(\nabla_\rho\nabla^\mu\varphi)\right], \quad (63)$$

where G_2, G_3, G_4 and G_5 are the function of φ and $X = -\nabla_\mu\varphi\nabla^\mu\varphi/2$. This four-dimensional action represents all single scalar field models whose equations of motion are kept up to second order in derivatives of the metric and of the scalar fields. As such, this action is general enough to encompass scalar-tensor models, although it is usually invoked to study the phenomenology of derivative couplings of φ.

VIII. BRANEWORLD SCENARIO

The idea of exploring higher dimensional spacetimes is another possible way of going beyond GR. Perhaps the simplest and most popular realization of this idea is that of the braneworld Dvali-Gabadadze-Porrati (DGP) model.

$$S = \int d^4x\,\sqrt{-g}\left[\frac{R}{16\pi G} - \mathcal{L}_{\rm matter}(\psi, g_{\mu\nu})\right] + \int d^5x\,\sqrt{-g^{(5)}}\frac{R^{(5)}}{16\pi G^{(5)}}. \quad (64)$$

where the superscript (5) indicates quantities defined w.r.t. the 5D metric $g^{(5)}_{\mu\nu}$.

IX. MAGNETIC FIELDS IN THE MODIFIED GRAVITY THEORY

will write it later....

X. CONFORMAL TRANSFORMATION

XI. QUANTUM FLUCTUATIONS OF GENERIC MASSLESS SCALAR FIELDS IN CURVED SPACE

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